Title	Compact Moduli Spaces of Kähler–Einstein Fano Varieties
Author(s)	Odaka, Yuji
Citation	Publications of the Research Institute for Mathematical Sciences (2015), 51(3): 549-565
Issue Date	2015
URL	http://hdl.handle.net/2433/202112
Right	© 2015 EMS Publishing House.; This PDF is the author's version of the work (Yuji Odaka. Compact Moduli Spaces of K ähler–Einstein Fano Varieties. PUBLICATIONS OF THE RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES Volume 51, Issue 3, 2015, pp. 549–565)
Type	Journal Article
Textversion	author

COMPACT MODULI SPACES OF KÄHLER-EINSTEIN FANO VARIETIES

YUJI ODAKA

Abstract. We construct geometrically compactified moduli spaces of Kähler-Einstein Fano manifolds.

1. Introduction

In this paper, we construct compactified moduli algebraic spaces of Fano manifolds which have Kähler-Einstein metrics or equivalently (thanks to [CDS], [Tia2], combined with [Ber], [Mab1],[Mab2]) are K-polystable, following the (precise) conjecture in [OSS] formulated with C. Spotti and S. Sun. The K-stability was originally introduced by G. Tian [Tia] and formulated in a purely algebraic way by S. Donaldson [Don0]. Brief explanations of the definition and the statement of the recent equivalence theorem with Kähler-Einstein metrics existence are given at the beginning of section 2 and the subsection 3.2. Roughly speaking, our main result of this paper is:

Theorem 1.1 (Algebro-geometric statement, over $\mathbb{C}$). *For any positive integer n, there is a "canonical" algebraic compactification $\bar{M}$ of the moduli space M of K-polystable smooth Fano manifolds of dimension n, whose boundary paramterises K-polystable (kawamata-log-terminal $\mathbb{Q}$-Gorenstein smoothable) $\mathbb{Q}$-Fano varieties of the same dimension.*

More precisely speaking, the compactification $\bar{M}$ is an algebraic space in the sense of Artin [Art] in the above result. For most precise meanings, see (section 2 and) Theorem 2.3. We further expect the compactification to be a *projective scheme*, following the idea of Fujiki-Schumacher [FS]. See the precise expectation in [OSS, subsections 3.4, 6.2] or our section 2 (which follows [OSS]).

Date: Communicated by T. Mochizuki. Received December 22, 2014. February 22, 2015. March 19, 2015.

2010 *Mathematics Subject Classification.* Primary 14K10; Secondary 33-00, 14J45, 14D20.

Key words and phrases. Moduli algebraic stack, Fano manifolds, Kahler-Einstein metrics, Gromov-Hausdorff limits.

The corresponding complex differential geometric (roughest) restatement of Theorem 1.1 is the following.

Theorem 1.2 (Differential geometric re-statement)**.** *The Gromov-Hausdorff compactification of the moduli space of Kähler-Einstein (smooth) Fano manifolds has a structure of compact Hausdorff Moishezon analytic space.*

This compactification extends that of the explicit 2-dimensional case study in [OSS], which was previously and pioneeringly proved in the case of complete intersection of 2 quadric 3-folds (i.e. degree 4 del Pezzo surfaces) in the old work of Mabuchi-Mukai [MM] much before the introduction of K-stability.

This contrasts to the "canonically polarised"case (i.e. of ample canonical class) the idea which (for dimensions higher than 1) goes back to Shepherd-Barron [SB]. This case was systematically studied by Kollár-Shepherd-Barron [KSB] for surfaces, extended by Alexeev to higher dimension [Ale], and now being accomplished with technical details (a book by Professor Kollár [Kol] with all the details are being expected to appear). Coining the main contributors to the construction in their honors, that theory is often attached with *Kollár-Sherpherd-Barron-Alexeev*, or with abbreviation "KSBA".

The novel difference is that in our case all the varieties parametrised are *normal* (even kawamata-log-terminal), hence irreducible, while KSBA degenerations are usually non-normal as even the simplest case - stable curve [DM] - can have up to $3g-3$ components.

However, in the meantime those two moduli compactifications can be seen in a unified point of view, i.e. as examples of moduli of K-(semi)stable varieties since the semi-log-canonical varieties of ample canonical class is also K-stable by [Od1] ("K-moduli" cf. e.g., [Od0, section 5], [Spo, Chapter 1]). Inspired by the breakthroughs [DS] and [Spo], in [OSS, Conjecture 6.2], the precise formulation of the K-moduli conjecture for Fano varieties case is worked out and we will quickly review a part of this in the next section.

A key technical result may be interesting of its own. That is, we will establish the following deformation picture. The (easier) half of the following statement is proved in [OSS] and the rest is essentially depending on [LWX], [SSY] which in turn use the idea of *Donaldson's continuity method* [CDS], [Tia2]. Our statement is as follows, but we again leave the detailed statement to Theorem 3.2.

Theorem 1.3. *If a Kähler-Einstein $\mathbb{Q}$-Fano variety X is $\mathbb{Q}$-Gorenstein smoothable, then in a local $\mathbb{Q}$-Gorenstein (Kuranishi) deformation* *space of X which we denote by $Def(X)$, the existence of Kähler-Einstein metric on the corresponding $\mathbb{Q}$-Fano variety is equivalent to the GIT polystability of the $Aut(X)$-action on $Def(X)$.*

As we mentioned, we have already proved in [OSS, Lemma 3.6] that the classical GIT polystability of points corresponds to Kähler-Einstein $\mathbb{Q}$-Fano varieties, which is the easier half of the above theorem 1.3. This extends the picture of [Tia], [Don1] for the commonly studied "Mukai-Umemura 3-fold" case, and the general result by Szekelyhidi [Sze] which depends on the infinite dimensional implicit function theorem. Our proof essentially depends on the recent development for one-parameter deformations cases in [LWX] and [SSY]. We expect that the $\mathbb{Q}$-Gorenstein smoothability condition is unnecessary but do not know how to prove in that generality, by current technologies. It is also related to the list of questions for furture in the final section.

Actually many of the main technical ingredients of the proof are mostly already in previous papers in this several years i.e. [DS], [Spo], [Od2], [OSS] and recent [SSY], [LWX] and this paper would not claim elaboration of the essential ideas from before.

Acknowledgements. This paper originally grew out from much more personal and incomplete notes sent to and shared with Cristiano Spotti, Song Sun, Chengjian Yao from October 2014 that is three months after when the results of [SSY] were informed to the author. It was in July of 2014 during the visit of S.Sun to Kyoto and Tokyo, and also there were several seminar talks made by them some months before the appearance of [SSY]. The author is grateful to all of their neat clarification about their results as well as their helpful comments on the draft, and would like to say that they also made essential contributions partially through [SSY] (and [OSS], [DS]). We also thank Jarod Alper for his kind communications about sub-section 3.1.

When the author started to expect "K-moduli" [Od0, section 5], he struggled but could never imagine how to prove even in Fano case and the partial proof obtained here just makes clear that he is watching the beauty "*on the shoulder of (modern) Giants*", especially for the case of this paper as we do not bring any essentially new idea but simply combining the circle of ideas and some standard arguments. I would like to take this opportunity to thank all the professors, colleagues and friends for the tutorials.

While finishing the first manuscript of this paper, the author learnt the possibility of partial overlap with the revision (to their 2nd version) of [LWX] and our paper. We would like to clarify that we worked out independently and both results appear on the same day on the internet.

Finally the author acknowledges the partial support by JSPS Kakenhi 30700356 (Wakate (B)).

2. Precise formulation of K-moduli

In this section, we put the precise formulation of the K-moduli. Before that, let us recall that the K-stability of a $\mathbb{Q}$-Fano variety X is, roughly speaking, defined as positivity of all the Donaldson-Futaki invariants (a variant of the GIT weight) associated to every one parameter isotrivial degenerations of X. We put more precision later in the subsection 3.2. The recent development shows the following.

Theorem 2.1 ([CDS], [Tia2] for smooth X, [SSY] for singular X). *For any $\mathbb{Q}$-Gorenstein smoothable klt $\mathbb{Q}$-Fano variety X, the existence of Kähler-Einstein metric is equivalent to the K-polystability of X.*

For the definition of Kähler-Einstein metrics on singular klt (kawamata-log-terminal) $\mathbb{Q}$-Fano varieties, we refer to [Ber] or [SSY] for instance.

Now we explain our precise statement of the K-moduli existence, partially recalling [OSS]. The precision on local deformation picture will be put only at the final section (Theorem 3.2).

For partial self-containedness and the convenience for the readers, we recall the notion of *KE moduli stack*, introduced for algebraically oriented people. We also note that in [OSS], the notion of *KE analytic moduli spaces* (for analytic oriented people) was introduced as well. For the general theory of algebraic stack, we would like to refer to textbooks such as [LM].

For those who are not familiar with stacky language, we note that algebraic stack (appearing here) is more or less an algebraic scheme (such as Hilbert scheme, Chow variety) attached with "glueing data" which identifies points on the scheme which "parametrises the same objects". Artin stack is the most general category of algebraic stack, allowing "non-discrete automorphism groups" of the parametrised objects, while Deligne-Mumford stack is, roughly speaking, for those objects with only discrete automorphism groups. The point of introduction of stacky language here is, more or less, to make the statement of most precise form with the information on flat families of Fano varieties (in concern with Kähler-Einstein metrics).

Definition 2.2 ([OSS, Definition 3.13]). A moduli algebraic (Artin) stack $\bar{\mathcal{M}}$ of $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties is called a *KE moduli stack* if

(i) there is a categorical moduli algebraic space $\bar{M}$

(ii) it has an étale covering $\{[U_i/G_i]_i\}$ of $\bar{\mathcal{M}}$ where U_i is affine algebraic scheme and G_i is some reductive algebraic group, on which there is some G_i-equivariant $\mathbb{Q}$-Gorenstein flat family of $\mathbb{Q}$-Fano varieties.

(iii) Closed G_i-orbits in U_i parametrize $\mathbb{Q}$-Gorenstein smoothable Kähler-Einstein $\mathbb{Q}$-Fano varieties via the families of (ii), and via the canonical map $\varphi_i \colon U_i \to \bar{M}$, each such orbit maps to a closed point of $\bar{M}$ and every closed point of $\bar{M}$ can be obtained in this manner for some i.

We call the coarse algebraic space $\bar{M}$ of (i) a *KE moduli space*. If it is an algebraic variety, we also call it *KE moduli variety.*

Recall that $\bar{M}$ is the *coarse moduli algebraic space* of the Artin stack $\bar{\mathcal{M}}$ means that there is a morphism $\bar{\mathcal{M}} \to \bar{M}$ and it is universal among the morphisms from $\bar{\mathcal{M}}$ to algebraic spaces. In our case, thanks to the condition (ii) and (iii) $\bar{M}$ is also set-theoritically "nice" i.e. bijectively corrsponds to Kähler-Einstein $\mathbb{Q}$-Fano varieties.

For the definition of more differential geometric version "KE analytic moduli space", we refer to [OSS, Definition 3.14, 3.15] since we do not use the notion in this paper and moreover it naturally follows from our construction in this paper that $\bar{M}$ satisfies the defining conditions of the notion.

In this paper, we prove the Conjecture 6.2 in [OSS] in $\mathbb{Q}$-Gorenstein smoothable case, i.e. those which contain the moduli of smooth Fano manifolds.

Theorem 2.3 (Refined statement of the K-moduli existence)**.** *We fix the dimension of $\mathbb{Q}$-Fano varities in concern, as n. There is a KE moduli stack $\bar{\mathcal{M}}^{GH}$, in the sense of* [OSS]*. In particular, $\bar{\mathcal{M}}^{GH}$ has a coarse moduli algebraic space $\bar{M}$ as a proper separated algebraic space, and $\bar{\mathcal{M}}^{GH}$ is good in the sense of Alper* [Alp].

Then from the Gromov-Hausdorff compactification M^{GH} (in the sense of [DS], [OSS]*), which is a priori just a compact Hausdorff metric space, there is a homeomorphism*

$$\Phi \colon \bar{M}^{GH} \to \bar{M},$$

such that $[X]$ and $\Phi([X])$ parametrize isomorphic $\mathbb{Q}$-Fano varieties for any $[X] \in \bar{M}^{GH}$.

We remark that the above "Gromov-Hausdorff" is in the refined sense, that is, with care of complex (algebraic) structures as defined and explained in [DS], [SSY] etc.

3. Proof of the main theorems

3.1. **Affine étale slice in the Hilbert scheme.** We begin the proof of our Main theorem 2.3, which will be completed in the end of subsection 3.3. In this subsection, we construct an affine slice around $[X]$ inside appropriate Hilbert scheme, where X is the $\mathbb{Q}$-Fano variety in concern. In the next subsection, using that slice, we formulate and prove the local deformation picture of Kähler-Einstein metrics.

We fix the dimension n of the Fano varieties in concern, and consider a finite disjoint union of components of the Hilbert scheme, which we denote by *Hilb*, which includes all smooth Kähler-Einstein Fano manifolds of dimension n and its Gromov-Hausdorff limits. Such finite type *Hilb* exists thanks to the recent breakthrough by Donaldson-Sun [DS] and the "classical" boundedness result by Kollár-Miyaoka-Mori [KMM]. In [DS], it is even proved that we can assume that they are all m-pluri-anticanonically embedded inside $\mathbb{P}^N$ with some uniform exponent m and $N = h^0(-mK_X) - 1$. We work in this setting so that our construction a priori depends on m but we do not expect so (see the remark 3.5 which we put in our revision).

We set $Hilb^{KE}$ as those which parameterize all m-pluri-anticanonically embedded Kähler-Einstein $\mathbb{Q}$-Fano varieities. Obviously $Hilb^{KE}$ is an $SL(N+1)$- invariant (equivalently, $PGL(N+1)$-invariant) subset of *Hilb* but note that it does not have a scheme structure in general. In fact, as we will show in the next subsection 3.3 without using the results in this subsection, $Hilb^{KE}$ is a *constructible* subset in *Hilb*. So from now on, we replace *Hilb* by the Zariski closure of $Hilb^{KE}$ so that we can assume that $Hilb^{KE}$ is dense inside *Hilb*. From now on, we work inside this replaced *Hilb*.

Take any point $[X] \in Hilb^{KE}$. From [CDS, III Theorem 4], an extension of the Matsushima's theorem [Mat], we know that the automorphism group $Aut(X)$ is a reductive algebraic group. Note that $Aut(X)$ is the isotropy (stabiliser) subgroup of the natural PGL-action on *Hilb*. Thus the isotropy subgroup of SL-action on *Hilb*, which we denote as $\tilde{Aut}(X)$, is a central extension of $Aut(X)$ by μ_{N+1}, the finite group of $(N+1)$-th roots of unity isomorphic to $\mathbb{Z}/(N+1)\mathbb{Z}$ which acts trivially on *Hilb*. The reason why we think also SL-action not only PGL-action is sometimes it is needed to make the action available at the level of vector space $H^0(X, -mK_X)$ i.e. cone over the projective space $\mathbb{P}^N$.

Also let us recall that $Hilb \subset \mathbb{P}_*(V)$ with some SL-representation V from the construction of the Hilbert scheme by Grothendieck. (Here $\mathbb{P}_*$ denotes covariant projectivisation unlike Grothendieck's notation.

) Noting that $[X]$ corresponds to a $\tilde{Aut}(X)$-invariant one dimensional vector space $\mathbb{C}v \subset V$, we can decompose $\tilde{Aut}(X)$'s linear representation as $V = \mathbb{C}v \oplus V'$ where V' is also $\tilde{Aut}(X)$-invariant. We owe Jarod Alper for the clarification about this and be grateful to him. This is possible since we know $\tilde{Aut}(X)$ is reductive. Then we can take an $Aut(X)$-invariant open subset $U_{[X]}$ as $Hilb \setminus \mathbb{P}_*(V')$. This is also affine since $\mathbb{P}_*(V')$ is an ample divisor of the original projective space $\mathbb{P}_*(V)$.

Note that this open neighborhood $U_{[X]}$ of $[X]$ is only $Aut(X)$-invariant (or equivalently $\tilde{Aut}(X)$-invariant), but *not* necessarily SL-invariant. In the meantime, this affine-ness of $U_{[X]}$ enables us to apply the following techniques of taking étale slice mainly due to [Luna] (a.k.a., Luna's "*étale slice theorem*" cf. [Dre, 5.3]). We include the short outline of the proof for the readers' convenience, partially because we slightly extend original theorem of [Luna], but basically the argument below is from the nice exposition of [Dre] on the Luna's theory [Luna]. First we can easily construct a closed immersion of $U_{[X]}$ into an $Aut(X)$-acted smooth affine space $\tilde{U}_{[X]}$ (cf. e.g, [Dre, Lemma 5.2]) with the same embedded dimension of $[X] \in U_{[X]}$. Then for the proof of étale slice theorem of [Luna] (cf., e.g., [Dre, Lemma 5.1]), it is proved that there is an $Aut(X)$-equivariant affine regular map $\varphi\colon \tilde{U}_{[X]} \to (T_{[X]}U_{[X]})$ which is étale at $[X]$. This is again depending on the reducitivity of $Aut(X)$. We use this equivariant map as follows.

We decompose the $Aut(X)$-representation $T_{[X]}U_{[X]}$ as $T_{[X]}(SL(N+1)[X] \cap U_{[X]}) \oplus N$ with some $Aut(X)$-invariant subvector space N. Then we define $V_{[X]} := (\varphi^{-1}N \cap U_{[X]}) \subset U_{[X]}$, which is an $Aut(X)$-invariant locally closed affine subset of $Hilb$ including $[X]$. Then this $V_{[X]} \subset U_{[X]}$ is an étale slice in the sense of [Luna, Dre], in particular $[V_{[X]}/Aut(X)] \to [U_{[X]}/PGL]$ is an étale morphism (between two quotients stacks). We omit more details and the rest of the proof of this known fact since it simply follows from the proof of [Dre, Theorem 5.3] or [AK, subsection 2.2].

3.2. **K-stability via CM line bundle.** Before proceeding to next arguments, we briefly recall the fundamental relation of the K-stability and *the CM line bundle* ([FS, PT]), which we regard as a definition of the K-stability in this paper.

The CM line bundle, in our setting, is a certain SL-equivariant line bundle λ_{CM} on $Hilb$ ([FS], [PT], [FR]). As the actual construction is a little complicated and we do not need in this paper, we omit its details and refer to [FR].

In our setting, for the given positive integer parameter m, the $K_{(m)}$-stability of $\mathbb{Q}$-Fano varieties means the following (as in [Od2], just following [Don0]).

Definition 3.1. As in the previous subsection, suppose that a (klt) $\mathbb{Q}$-Fano variety X satisfies that $-mK_X$ is a very ample line bundle ($m \in \mathbb{Z}_{>0}$). Then the $\mathbb{Q}$-Fano variety X (more precisely, $(X, -K_X)$) is said to be $K_{(m)}$-stable if for any nontrivial one parameter subgroup $f\colon \mathbb{C}^* \to SL$, minus the weight of $\lambda_{CM}|_{lim_{t\to 0}(f(t)[X])}$ (called as the Donaldson-Futaki invariant associated to f) is positive. The one parameter degeneration of X along $\overline{f(\mathbb{C}^*)\cdot [X]} \subset \mathit{Hilb}$ is called "*test configuraiton*" by [Don0].)

Similarly, X is said to be $K_{(m)}$-semistable (resp. $K_{(m)}$-polystable) if all the Donaldson-Futaki invariants are non-negative (resp. X is semistable and the Donaldson-Futaki invariant of f is positive if and only if the orbit closure $\overline{f(\mathbb{C}^*)\cdot [X]} \subset \mathit{Hilb}$ is contained in the SL-orbit of X (such a degeneration is called "*product* test configuration")).

X is said to be K-stable (resp. K-semistable, K-polystable) if it is $K_{(m)}$-stable (resp. $K_{(m)}$-semistable, $K_{(m)}$-polystable) for all sufficiently divisible positive integer m.

3.3. **Local GIT polystability.** In this subsection, we apply [OSS, Lemma 3.6] to the $Aut(X)$-action on the Affine étale slice $V_{[X]}$ and see that

> the points corresponding to some Kähler-Einstein $\mathbb{Q}$-Fano varieties are GIT polystable in $V_{[X]}$ with respect to the $Aut(X)$-action,

and we denote the polystable locus in $V_{[X]}$ as $V_{[X]}^{ps}$. The following theorem shows that the converse to [OSS, Lemma 3.6] also holds in appropriate sense, and later on this will be crucial for us.

Theorem 3.2 (Local deformation picture of KE Fano varieties)**.** *For small enough affine étale slice $V_{[X]}$, i.e. after shriking $V_{[X]}$ to $Aut(X)$-invariant open affine neighborhood of $[X]$ if necessary, we have*

$$V_{[X]}^{ps} = V_{[X]} \cap \mathit{Hilb}^{KE}.$$

Recall that $V_{[X]}^{ps}$ denotes the GIT poly-stable locus of the affine slice $V_{[X]}$ in the Hilbert scheme, with respect to the $Aut(X)$-action.

It roughly says that, étale locally, the existence of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties is equivalent to the classical GIT polystability, at least in the $\mathbb{Q}$-Gorenstein smoothable case (we expect this is the case in non-smoothable case as well). Note that the above statement is about "local" deformation picture in the sense we need to shrink $V_{[X]}$ in general. Otherwise the statement is false and indeed the proof requires that shrinking.

This refines [Tia, section 7], [Don1, subsection 5.3] which treated Mukai-Umemura (Fano) 3-folds, $\mathbb{Q}$-Fano varieties case of [Sze] and of course [OSS, Lemma 3.6]. We expect that this will be a fundamental tool in the further study of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties in future.

proof of Theorem 3.2. The one side that $V_{[X]} \cap Hilb^{KE} \subset V_{[X]}^{ps}$ is exactly (a special case of) [OSS, Lemma 3.6] and here is the argument for the other side i.e.

$$V_{[X]}^{ps} \subset V_{[X]} \cap Hilb^{KE}.$$

We prove that this holds, once we replace $V_{[X]}$ with small enough affine $Aut(X)$-invariant slice if necessary.

Note that the difference set $V_{[X]}^{ps} \setminus (V_{[X]} \cap Hilb^{KE})$ is constructible since both $V_{[X]}^{ps}$ and $(V_{[X]} \cap Hilb^{KE})$ are constructible subsets. The constructibility of polystable locus is a standard fact in the Geometric Invariant Theory. We now explain how to show the constructibility of $(V_{[X]} \cap Hilb^{KE}) \subset V_{[X]}$. Indeed, due to [SSY, Theorem 1], we know the equivalence of K-polystability and existence of Kähler-Einstein metrics for $\mathbb{Q}$-Gorenstein smoothable Fano varieties in general. Moreover, combining [CDS, esp. II Theorem1, III Theorem 2], [SSY, 4.2.2] and the arguments of [Od2, esp. (2.4-8)], we know that it is also equivalent to the quantised "$K_{(m)}$-polystability" in the above sense of subsection 3.2 for sufficiently divisible uniform $m \gg 0$ i.e. we can bound the exponent m for testing K-(poly)stability. For the readers' convenience, we recall from [Od2, esp. (2.4-8)] that the main point of the uniform bound m was the uniform positive lower bounds of (small) *angles* of conical Kähler-Einstein metrics on all the $\mathbb{Q}$-Fano varieties parametrised in *Hilb*.

Then the proof of the constructibility of the $K_{(m)}$-polystable locus inside *Hilb* follows from the arguments in [Od2, esp. (2.10-12)] only with the additional but simple concern whether the test configurations are of product type or not.

To prove the theorem, we suppose the contrary and get contradiction. So let us suppose that *for any small enough* affine $Aut(X)$-invariant slice $V_{[X]}$ of $[X]$, we have $V_{[X]}^{ps} \neq V_{[X]} \cap Hilb^{KE}$.

Therefore, from our assumption that $V_{[X]}^{ps} \neq V_{[X]} \cap Hilb^{KE}$, we have an irreducible locally closed subvariety W inside the difference subset

$V^{ps}_{[X]} \setminus (V_{[X]} \cap \mathit{Hilb}^{KE})$ whose closure meets $[X]$ and we take a sequence P_i in W converging to $[X]$. Otherwise, we can shrink $V_{[X]}$ to make it satisfies $V^{ps}_{[X]} = V_{[X]} \cap \mathit{Hilb}^{KE}$. Now we fixed our slice $V_{[X]}$.

We take any SL-equivariant compactification of the algebraic group SL (such as [DP], or apply [Sum]) and denote it with $\bar{SL}$ and consider the rational map $\varphi\colon \overline{V_{[X]}} \times \bar{SL} \dashrightarrow \mathit{Hilb}$ induced by the SL-action. Here $\overline{V_{[X]}}$ denotes the Zariski closure of $V_{[X]}$ inside Hilb. Then we take a SL-equivariant resolution of indeterminancy of φ as

$$\tilde{\varphi}\colon T \to \mathit{Hilb}.$$

So T is a certain SL-equivariant blow up of $\overline{V_{[X]}} \times \bar{SL}$ along some ideal co-supported on $\overline{V_{[X]}} \times (\bar{SL} \setminus SL)$. Via the morphism from T to Hilb, we can regard T as a parameter space of Fano varieties and its degenerations.

Then take sequences $P_i \in W \subset \overline{V_{[X]}} \simeq \overline{V_{[X]}} \times \{e\} \subset T (i = 1, 2, \cdots)$ which converges to $[X] \in V_{[X]}$ and $P_{i,j} \in V_{[X]} \simeq V_{[X]} \times \{e\} \subset T(i, j = 1, 2, \cdots)$, parametrising smooth Kähler-Einstein Fano manifolds $X_{i,j}$, which converges to P_i when j goes to infinity.

Thanks to [DS], we know that (by taking subsequence) the Gromov-Hausdorff limit of $X_{i,j}$ with Kähler-Einstein metrics exists as another Kähler-Einstein $\mathbb{Q}$-Fano variety Y_i. Furthermore, from their construction as a limit inside the Hilbert scheme (cf., [DS, Theorem 1.2]), we know that there is a sequence of elements of SL which we denote by $\phi_{i,j}$ such that $lim_{j\to\infty}\phi_{i,j}(P_{i,j})$ represents a point Q_i which parametrises the (m-th pluri-anticanonically embedded) Kähler-Einstein Fano variety Y_i, for each fixed i. By the standard diagonal argument, it also follows from [DS] that $lim^{GH}_{i\to\infty} Y_i$ exists (limit in the (refined) Gromov-Hausdorff sense as in [DS]) as yet another Kähler-Einstein $\mathbb{Q}$-Fano variety Y where the corresponding point will be denoted by $Q \in \mathit{Hilb}$. As the blow up morphism $T \to \overline{V_{[X]}} \times \bar{SL}$ is (topologically) a proper morphism, we can take all these points in T.

Our general idea is to apply (recently obtained) separated-ness theorem to the two "degenerations" of $X_{i,j}$ to $[X]$ and $[Y] = Q \in T$, both of which parametrise Kähler-Einstein $\mathbb{Q}$-Fano varieties. To put precision on the idea, from now on, we proceed to some more algebro-geometric arguments.

Set T^o as the (open dense) subset of T which is the preimage of $SL \subset \bar{SL}$. We also set $\partial T := T \setminus T^o$. Consider some general affine curve $C \subset T$ which passes through Q and intersects $\partial T \cup (V^{ps}_{[X]} \setminus (V_{[X]} \cap \mathit{Hilb}^{KE}))$ only at the point $\{Q\}$.

On the other hand, take the natural retraction $r\colon T^o \to \overline{V_{[X]}}$ induced by $SL \to \{e\}$ where $e \in SL$ is the unit of special linear group SL and partially complete $C'^0 := r(C \setminus \{Q\})$ naturally to C' with $i\colon C \simeq C'$. Note that from the construction, r also naturally extends to a morphism

$$\tilde{r}\colon T \to \mathit{Hilb}$$

from the whole T. Then from our construction, the image $i(Q)$ is nothing but the original $[X] \in \mathit{Hilb}$. We can see it as follows. Since i should preserve the image of $\tilde{r}$, $\tilde{r}(i(Q)) = \tilde{r}(Q)$ and that $\tilde{r}(Q) = \tilde{r}(lim_{i\to\infty}(Q_i)) = \tilde{r}(lim_{i\to\infty}(lim_{j\to\infty}(P_{i,j})) = lim_{i\to\infty}(\tilde{r}(P_i)) = [X]$. The last equality follows from our construction of P_i. (Here all the limit symbols are in the usual sense of analytic topology).

The crucial result we need from now on is the following. Although we do not have any contribution on it in this paper, we would like to recall the result as we need a comment (on how to combine [LWX],[SSY],[CDS], as written below) on the proof to make things rigorous. I thank S.Sun for the mathematical clarification of this point.

Theorem 3.3 ([LWX, Thm1.1 of v1]+[SSY, Thm1.1],[CDS])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two $\mathbb{Q}$-Gorenstein flat deformations of Kähler-Einstein $\mathbb{Q}$-Fano varieties over a smooth curve $C \ni 0$. Suppose $\mathcal{X}_t \cong \mathcal{Y}_t$ for $t \neq 0$ and further that these are all smooth (i.e. generically smooth). If $\mathcal{X}_0$ and $\mathcal{Y}_0$ are both K-polystable, then they are isomorphic $\mathbb{Q}$-Fano varieties.*

This follows from the combination of [LWX, v1] and [SSY, Theorem 1.1]. Note that for *separateness*, [SSY, Corollary 1.2] needs to assume that $\mathcal{X}_0$ and $\mathcal{Y}_0$ have discrete automorphism groups, while [LWX, Remark 6.11] needs to assume that $\mathcal{X}_0$ and $\mathcal{Y}_0$ have reductive automorphism groups. But from [SSY, Theorem 1.1] we know both $\mathcal{X}_0$ and $\mathcal{Y}_0$ admit KE metrics, so satisfy the assumption on *reductivity* of [LWX, v1] by [CDS, III, Theorem 4]. (The author had once attempted to prove this *separateness* with Professor Richard Thomas but the arguments had a technical gap.)

We apply the theorem above to the two families of $\mathbb{Q}$-Fano varieties corresponding to $C \subset T$ and $C' \subset T$. Then we can show that Q is in the SL-orbit of $[X] \in \mathit{Hilb}$, hence in T^o in particular. Recall that Q was defined as the limit of Q_i. Hence for $i \gg 0$, Q_i is also in T^o. Then it implies that by [OSS, Lemma 3.6], $i(Q_i) \in V_{[X]}$, which is well-defined, is GIT polystable with respect to the action of the automorphism group $Aut(X)$.

Then we get a contradiction from the general theory of Geometric Invariant Theory [GIT] since $i(Q_i)$ and P_i are both GIT polystable, while being the limits of sequences which parametrises the same polystable point. This completes the proof. □

Proposition 3.4. *Let X be an arbitrary $\mathbb{Q}$-Gorenstein smoothable Kähler-Einstein $\mathbb{Q}$-Fano variety and denote the corresponding point in the Hilbert scheme as $[X]$ which represents m-pluri-anticanonically embedding $[X] \in Hilb^{KE}$. Then there is a small enough affine $Aut(X)$-invariant slice $V_{[X]}$ of the natural PGL-action on Hilb such that an open neighborhood (in analytic topology) of $[\bar{X}]$ in the GIT (categorical) quotient $V_{[X]}//Aut(X)$ naturally maps homeomorphically to $\bar{M}^{GH}$ (which eventually becomes an étale algebraic morphism with the algebraic structure on the latter).*

Analytically speaking, this is equivalent to saying that there is an open subset W of $[X]$ in $V_{[X]}$ and an analytically open neighborhood N of $[X] \in \bar{M}^{GH}$ such that there is a natural homeomorphism

$$N \to (W \cap V^{ps}_{[X]})/Aut(X),$$

preserving the $\mathbb{Q}$-Fano varieties being parametrised.

proof of Proposition 3.4. The continuity from N to $(W \cap V^{ps}_{[X]})/SL$ follows from Donaldson-Sun [DS, (proof of) Theorem 1.2]. The quotient space $Hilb^{KE}/SL$ satisfies the Hausdorff axiom due to the separatedness theorem 3.3 proved by ([LWX]+[SSY]) while $\bar{M}^{GH}$ is compact due to the Gromov compactness theorem. It is a general theorem that continuous bijection from a compact topological space (now $\bar{M}^{GH}$) to a Hausdorff space (now $Hilb^{KE}/SL$) is automatically homeomorphism. □

Summarising the above discussions, we conclude the proof of our main theorem 2.3, the moduli construction, as follows.

proof of Theorem 2.3. For each $[X_i] \in Hilb^{KE}$, i.e. X_i is one of smooth Kähler-Einstein Fano n-dimensional manifolds or one of their Gromov-Hausdorff limits (hence $\mathbb{Q}$-Fano varieties with Kähler-Einstein metrics by [DS]), let us consider $V_{[X_i]}$ constructed in the subsection 3.1. We replace $V_{[X_i]}$ by its open $Aut(X_i)$-invariant open neighborhood, if necessary, to make it satisfy the requirement in Theorem 3.2.

Note that for each X_i, $PGL \cdot V_{[X_i]}$ is a Zariski open subset in *Hilb*. It follows from the fact that since we constructed $V_{[X_i]} \subset U_{[X_i]}$ as an étale slice, $PGL \times_{Aut(X_i)} V_{[X_i]} \to Hilb$ is an étale morphism so in particular an open morphism. Thus by quasi-compactness of *Hilb*, we only need finitely many i such sets $PGL \cdot V_{[X_i]}$ to cover $Hilb^{KE}$.

We note that $\varphi_i\colon [V_{[X_i]}/Aut(X_i)] \to [Hilb/PGL]$ is an étale morphism between two quotient stacks, since again the morphism $PGL \times_{Aut(X_i)}$

$V_{[X_i]} \to U_{[X_i]} \subset Hilb$ is strongly étale (in the sense of [Dre, subsection 1.1]). Please note that it is a priori *not* necessarily open immersion (of algebraic stacks) because the slice $V_{[X_i]}$ is just an *étale* slice. Glueing together $[V_{[X_i]}/Aut(X_i)]$ via φ_is which is by definition possible inside $[Hilb/PGL]$, we obtain $[W/PGL]$ with $W = \cup_i(PGL \cdot V_{[X_i]}) \subset Hilb$, a moduli Artin stack which we denote as $\bar{\mathcal{M}}$. Furthermore, as the property [Dre, subsection 1.1 (ii)] of the étale slice $V_{[X_i]}$ (cf., also [Dre, 5.3]) shows, categorical quotients $V_{[X_i]}/Aut(X_i)$ glue together to form a coarse moduli algebraic space $\bar{M}$ of the Artin stack $\bar{\mathcal{M}}$.

The fact that it is a KE moduli stack in the sense of Definition 2.2 ([OSS]) now follows from Theorem 3.2. Indeed the condition (iii) of Definition 2.2 is exactly the statement of Theorem 3.2 and we have proved the condition (i) of Definition 2.2 above. The remaining (ii) of (2.2), which says that the flat family on $V_{[X_i]}$ is $\mathbb{Q}$-Gorenstein flat family (once we shrink $V_{[X_i]}$ enough), can be easily checked as follows. (Please also see [OSS, (2.4)] for essentially the same arguments.) Actually in general if we have a point $[X]$ in *Hilb* corresponding to some normal variety X, its deformation parametrised in a neighborhood in *Hilb* is automatically $\mathbb{Q}$-Gorenstein deformation. We set the locus of *Hilb* which parametrises normal varieties as $Hilb_{\rm normal} \subset Hilb$, that is automatically open subset as it is well known. We denote its subset which parametrises singular (but normal) varieties as $Hilb_{\rm normal.singular}$. Let us take a log resolution of singularities of the pair $(Hilb_{\rm normal}, Hilb_{\rm normal.singular})$ after Hironaka, as $f\colon S \to Hilb$ so that $f^{-1}(Hilb_{\rm normal.singular})$ is a (simple normal crossing) Cartier divisor Σ of S. Then we have a flat projective family $\pi\colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \to S$ and

$$\mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X}\setminus\pi^{-1}(\Sigma)} \sim_{(S\setminus\Sigma)} \mathcal{O}_{(\mathcal{X}\setminus\pi^{-1}(\Sigma))}(-mK_{\mathcal{X}\setminus\pi^{-1}(\Sigma)}). \tag{1}$$

The above (1) implies that there are Weil divisors $\mathcal{D}, \mathcal{D}'$ of $\mathcal{X}$ with $\mathcal{O}_{\mathcal{X}}(\mathcal{D}) = \mathcal{O}_{\mathcal{X}}(1), \mathcal{O}_{\mathcal{X}}(\mathcal{D}') = \mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}})$ (the latter is only a reflexive sheaf), which satisfies that $\mathcal{D} - \mathcal{D}'$ supports on $\pi^{-1}(\Sigma)$. In the meantime, any (a priori Weil-)divisor supported on the central fiber is a pull back of (Cartier) divisor of S supported on Σ since all the fibers of π are irreducible now. Hence, we get $\mathcal{O}(1) \sim_C \mathcal{O}(-mK_{\mathcal{X}})$.

Furthermore, the subset $Hilb_{\rm klt}$ of $Hilb_{\rm normal.singular}$ which parametrises (kawamata-)log-terminal varieties is a Zariski open subset, which follows from the arguments of [AH, Appendix A] (even easier than that since we only treat normal varieties). In particular, $V_{[X_i]}$ only parametrises $\mathbb{Q}$-Fano varieties, since each variety parametrised in $V_{[X_i]}$ has some isotrivial degeneration to a variety parametrised in $V^{ps}_{[X_i]}$ which is automatically a $\mathbb{Q}$-Fano variety. Summarising up, we proved the assertion ((ii) of Definition 2.2).

The topological space structure part is proved in Proposition 3.4. Indeed, note that Proposition 3.4 shows that the Gromov-Hausdorff compactification $\bar{M}^{GH}$ is homeomorphic to the coarse moduli space $\bar{M}$ constructed above. In particular it shows $\bar{M}$ satiefies the Hausdorff second axiom (essentially follows from [CDS]+[LWX](v1)+[SSY] cf., Theorem 3.3). So we complete the proof of Theorem 2.3. □

Remark 3.5. *This remark is newly put in our revision which appears in the* 20*th of March, 2015.* Our construction of the moduli stacks $\bar{\mathcal{M}}$ and their coarse moduli spaces $\bar{M}$ a priori depend on the positive integer parameter m (please recall that we consider the m-th pluri-anti-canonical polarisation of the $\mathbb{Q}$-Fano varieties). However we strongly believe that they actually do *not* depend on the sufficiently divisible m. Indeed, we can prove it under the following two hypotheses. To the best of the author's knowledge (as of March, 2015) full proofs of the hypotheses below are not available yet, although the revision (2nd version) of [LWX] have partial affirmative results (cf., their section 7) in this direction.

(i) The K-semistability is an *open condition* for any $\mathbb{Q}$-Gorenstein flat projective family of $\mathbb{Q}$-Fano ($\mathbb{Q}$-Gorenstein smoothable) varieties.
(ii) For any ($\mathbb{Q}$-Gorenstein smoothable) K-semistable $\mathbb{Q}$-Fano variety, say X, it has a test configuration whose central fibre is a KE $\mathbb{Q}$-Fano variety Y (which is K-polystable by [Ber]).

Our proof of the desired m-independence of our moduli $\bar{\mathcal{M}}$ and $\bar{M}$, under the hypotheses, is simple as follows. The above hypotheses imply that W coincides exactly with the (open) locus of $\mathbb{Q}$-*Gorenstein smoothable K-semistable* $\mathbb{Q}$-*Fano varieties*, which we denote as $Hilb^{sss}$. We prove it as follows. Recall that each $\mathbb{Q}$-Fano variety corresponding to a point of W, isotrivially degenerates to a KE $\mathbb{Q}$-Fano variety parametrises in $Hilb^{KE}$ by our Theorem 3.2 and the standard GIT. That fact, combined with the first hypothesis (i) implies $W \subset Hilb^{sss}$. On the other hand, (ii) and [DS] (especially their uniform bound of "k") imply $Hilb^{sss} \subset W$ straightforwardly. Thus our KE moduli stack $\bar{\mathcal{M}}$, which is isomorphic to the quotient stack $[W/PGL]$ whose definition involved m, is exactly the moduli Artin stack of $\mathbb{Q}$-Gorenstein flat projective family of K-semistable $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano varieties of dimension n. It is this universality which automatically implies that the moduli stacks $\bar{\mathcal{M}}$ do not depend on the integer m. In particular, their coarse moduli spaces $\bar{M}$ also do not depend on m.

We also make a brief mathematical remark in this revision (March, 2015) for the readers' convenience, about the mathematical relation with the 2nd version of [LWX]. It is that the moduli space constructed in the 2nd version of [LWX] is the semi-normalisation of reduced subscheme of our moduli.

4. For future

It may be needless to mention but the author would like to note that there are quite a lot of interesting problems to do from now on the K-moduli of Fano varieties, and we list some main of them possibly with my personal biase. Most of them (perhaps other than Question 2) are natural and being shared among the community of this subject and we just write down for the record.

Question 1. *How about concrete examples of $\mathbb{Q}$-Fano varieties?*

As far as the author knows, the only fully settled case is [MM],[OSS] which are for ($\mathbb{Q}$-Gorenstein smoothable) Del Pezzo surfaces. The author would guess [OSS, Lemma 3.6] and our Theorem 3.2 will be one of the key tools for this direction. For example, the author is tempted to expect that many of the standard GIT moduli spaces of hypersurfaces, such as cubic 3-folds and 4-folds case ([All], [Laza], [Yok1], [Yok2]), are examples of our K-moduli spaces (cf., [OSS, Theorem 3.4 and subsection 4.2]). The last prediction is partially inspired by discussions with Julius Ross.

Question 2. *How to construct Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds (and the K-moduli construction) in purely algebraic way?*

It is natural to expect that the (refined) GH limit, in the sense of [DS],[OSS] etc, is simply equivalent to K-polystable limit and then, partially inspired by [LX], [Od2, last section] (etc), characterised by the minimality of the degree of (family version of) Donaldson-Futaki invariant. And we further expect that the construction will essentially need the idea and theory of the Minimal Model Program.

Question 3. *How about non-smoothable $\mathbb{Q}$-Fano varieties?*

This is the much more general case, morally about the moduli space all of whose members parametrise *singular* (log-terminal) $\mathbb{Q}$-Fano varieties. At this moment, we (and [SSY],[LWX] etc.) all heavily depend on the ($\mathbb{Q}$-Gorenstein) smoothability of Fano varieties in concern, in order to apply [CDS], [Tia2] which are for smooth Fano *manifolds*. But as many algebraically oriented people agree as they told, it is natural to expect the completely same picture for *general $\mathbb{Q}$-Fano varieties.*

Question 4. *What about the projectivity of our moduli space?*

The expectation is that "descended" $\mathbb{Q}$-line bundle from the CM line bundle [FS], [PT] explained (with the proof of descending phenomenon) at the end of [OSS], will be *ample* on the coarse compact moduli space $\bar{M}$, ensuring the projectivity. The expectation is based on the general *Weil-Petersson* metrics as in [FS]. Indeed by [FS], any compact analytic subset of the coarse moduli space of *smooth* KE Fano manifolds with *discrete* automorphism groups (constructed in [Od2]) is projective. However, to treat general case, there are two main technical difficulties which are the presence of non-discrete automorphism groups (involving K-*semi*stable varieties) and the log-terminal singularities.

(Added in the revision of March, 2015:) Two and a half months after when the first manuscript of this paper appears, [LWX2] made an announcement of a partial progress along this line and claimed the quasi-projectivity of the open locus M of $\bar{M}$.

References

[AH] D. Abramovich, B. Hassett, Stable varieties with a twist, arXiv:0904.2797 (2009).

[Ale] V. Alexeev, Log canonical singularities and complete moduli of stable pairs, arXiv:9608013 (1996).

[All] D. Allcock, The moduli space of cubic threefolds, J. Algebraic Geom. (2003).

[Alp] J. Alper, On the local structure of Artin stacks, Journal of Pure and Applied Algebra, (2010).

[AK] J. Alper, A. Kresch, Equivariant versal deformations of semistable curves, preprint.

[Art] M. Artin, Algebraic spaces, Yale Mathematical Monographs vol. 3, Yale University Press (1971).

[Ber] R. Berman, K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics, arXiv:1205.6214 (2012).

[CDS] X. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds I,II,III. J. A. M. S. (2015).

[DP] C. DeConcini, C. Procesi, Complete symmetric spaces, Springer-Verlag Lecture Notes in Mathematics vol. 996, (1983).

[DM] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Publ. I. H. E. S. (1969).

[Don0] S. Donaldson, Scalar curvature and stability of toric varieties, J. Diff. Geom (2002).

[Don1] S. Donaldson, Kahler geometry on toric manifolds, and some other manifolds with large symmetry, arXiv:0803.0985 (2008).

[Don2] S. Donaldson, Algebraic families of constant scalar curvature Kähler metrics, arXiv:1503.05174. (cf. also the record of his talk available at `http://www.math.u-psud.fr/~repsurf/ERC/Bismutfest/Donaldson.PDF`)

[Dre] J-M. Drézet, Luna's slice theorem and applications, Algebraic group actions and quotients, 39-89, Hindawi Publ. Corp., Cairo, (2004).

[DS] S. Donaldson, S. Sun, Gromov-Hausdorff limits of Kahler manifolds and algebraic geometry, Acta Math. (2014).

[FR] J. Fine, J. Ross, A note on the positivity of CM line bundle, I. M. R. N. (2006).

[FS] A. Fujiki, G. Schumacher, The moduli space of extremal compact Ka ¨ hler manifolds and generalized Weil-Petersson metrics, Publ. R. I. M. S. (1990).

[GIT] D. Mumford, Geometric Invariant Theory (book), Springer-Verlag (1965).

[Kol] J. Kollár, A book in preparation.

[KSB] J. Kollár, N. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. (1988).

[KMM] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Diff. Geom. (1992).

[LM] G. Laumon, L. Moret-Bailly. Champs algébriques, Springer-Verlag (2000).

[Laza] R. Laza, Moduli space of cubic fourfolds, J. Algebraic Geom. vol. 18 (2009).

[Luna] D. Luna, Slices étales, Sur les groupes algébriques, Bull. Soc. Math. France (1973).

[LWX] C. Li, X. Wang, C. Xu, Degeneration of Fano Kähler-Einstein manifolds, arXiv:1411.0761**v1** (2014).

[LWX2] C. Li, X. Wang, C. Xu, Quasi-projectivity of the moduli space of smooth Kähler-Einstein Fano manifolds, arXiv:1502.06532 (2015).

[LX] C. Li, C. Xu, Special test configurations and K-stability of Fano varieties, Ann. of Math. (2014).

[MM] T. Mabuchi, S. Mukai, Stability and Einstein-Kähler metric of a quartic del Pezzo surface, (1993). Proceeding of 27th Taniguchi sympsium at 1990.

[Mab1] T. Mabuchi, K-stability of constant scalar curvature polarization, arXiv:0812.4903 (2008).

[Mab2] T. Mabuchi, A stronger concept of K-stability, arXiv:0910.4617 (2009).

[Mat] Y. Matsushima, Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne, Nagoya Math. vol. **11** 145-150, (1957).

[Od0] Y. Odaka, On the GIT stability of Polarised Varieties - a survey -, Proceeding of Kinosaki symposium (2010). available at his website.

[Od1] Y. Odaka, The Calabi conjecture and K-stability, I. M. R. N (2012).

[Od2] Y. Odaka, On the moduli of Kähler-Einstein Fano manifolds, Proceeding of Kinosaki symposium (2013), available at arXiv:1211.4833

[OSS] Y. Odaka, C. Spotti, S. Sun, Compact moduli space of Del Pezzo surfaces and Kähler-Einstein metrics, to appear in J. Diff. Geom. arXiv:1210.0858.

[PT] S. Paul, G. Tian, CM stability and the generalized Futaki invariant I, arXiv:0605278 (2006).

[SB] N-I. Sherpherd-Barron, Degeneration with numerically effective canoni- cal divisor, The Birational Geometry of Degenerations, Birkhauser, (1983).

[Spo] C. Spotti, Degenerations of Kähler-Einstein Fano manifolds, Ph.D thesis supervised by S.Donaldson (Imperial college London). available at arXiv:1211.5334.

[SSY] C. Spotti, S. Sun, C. Yao, Existence and deformations of Kähler-Einstein metrics on smoothable $\mathbb{Q}$-Fano varieties, arXiv:1411.1725 (2014).

[Sum] H. Sumihiro, Equivariant completions I, J. Math. Kyoto Univ. (1974).

[Sze] G. Szekelyhidi, The Kähler-Ricci flow and K-polystability, Am. J. Math. vol. 132 (2010).

[Tia] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. (1997).

[Tia2] G. Tian, K-stability and Kähler-Einstein metrics, arXiv:1211.4669 (2012).

[Yok1] M. Yokoyama, Stability of cubic 3-folds, Tokyo J. Math., 25, (2002).

[Yok2] M. Yokoyama, Stability of cubic hypersurfaces of dimension 4, RIMS Kokyuroku Bessatsu B9 (2008). available at `http://www.kurims.kyoto-u.ac.jp/ kenkyubu/bessatsu-j-back.html`

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `yodaka@math.kyoto-u.ac.jp`